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On M-Algebras, the Quantisation of Nambu-Mechanics, and Volume Preserving Diffeomorphisms

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Abstract: M-branes are related to theories on function spaces $\mathcal{A}$ involving M-linear non-commutative maps from $\mathcal{A} \times \cdots \times \mathcal{A}$ to $\mathcal{A}$. While the Lie-symmetry-algebra of volume preserving diffeomorphisms of T^M cannot be deformed when $M > 2$, the arising M-algebras naturally relate to Nambu's generalisation of Hamiltonian mechanics, e.g. by providing a representation of the canonical M-commutation relations, $[J_1, \cdots, J_M] = i\hbar$. Concerning multidimensional integrability, an important generalisation of Lax-pairs is given.

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1. Introduction

Generalizing fundamental concepts, such as Lie algebras or Hamiltonian dynamics, may have quite divers merits; it can lead to new, interesting possibilities, – or reassure oneself of our present notions. While the result that volume preserving diffeomorphisms of toroidal M-branes, as a Lie-symmetry algebra, cannot be deformed (if $M > 2$) is of the latter nature – the following ideas appear to be worthwhile persueing:
— Using a $*M$-deformation of the algebra of functions on some M-dimensional manifold for representing the M-linear analogue to Heisenberg's commutation relations that Nambu [1] encountered in multi-Hamiltonian dynamics.
— Generalizing the Jacobi identity for Lie algebras to a (2-bracket) identity involving $2M - 1$ elements of a vectorspace V for which an antisymmetric M-linear map (M-commutator) from $V \times \cdots \times V$ to V is defined (in a dynamical context, an identity involving M, rather than 2, of the M-commutators, may be preferred).
— A potential relevance of M-algebras to the quantisation of space-time.
Perhaps most importantly (on a concrete, practical level), an explicit example is given (the multidimensional diffeomorphism-invariant integrable field theories found in [2]) for the usefulness (envisaged some time ago [3]) of generalizing Lax-pairs to -triples, ...

2. M-algebras from M-branes

A relativistic M-brane moving in D-dimensional space time may be described, in a light-cone gauge, by the VDiffΣ-invariant sector of ([4])

$$H = \frac{1}{2} \int_\Sigma \frac{d^M\varphi}{\rho(\varphi)} (\vec{p}^{\,2} + g) \tag{1}$$

where g is the determinant of the M×M matrix $(g_{rs}) := (\nabla_r x^i \nabla_s x_i)_{r,s=1\cdots M}$, x^i and p_i $(i = 1, \cdots, D-2 =: d)$ are canonically conjugate fields, and ρ is a fixed non-dynamical density on the M-dimensional parameter-manifold Σ ($M = 1$ for strings, $M = 2$ for membranes,...). Generators of VDiffΣ, the group of volume-preserving diffeomorphisms of Σ (resp. the component connected to the identity), are represented by

$$K := \int_\Sigma f^r\, p_i\, \partial_r\, x^i\, d^M\, \varphi \tag{2}$$

with $\nabla_r f^r = 0$. g may be written as

$$g = \sum_{i_1 < i_2 < \cdots < i_M} \{x_{i_1}, \cdots, x_{i_M}\}\{x^{i_1}, \cdots, x^{i_M}\}, \tag{3}$$

where the 'Nambu-bracket' $\{\cdots\}$ is defined for functions $f_1, \cdots, f_M$ on Σ as

$$\{f_1, \cdots, f_M\} := \epsilon^{r_1 \cdots r_M}\, \partial_{r_1}\, f_1 \cdots \partial_{r_M}\, f_M. \tag{4}$$

This trivial, but important observation suggests to consider Hamiltonians

$$H_\lambda := \frac{1}{2} Tr\Big(\vec{P}^{\,2} \pm \sum_{i_1<\cdots<i_M} [X_{i_1},\cdots,X_{i_M}]^2_\lambda\Big), \tag{5}$$

resp.

$$H_\lambda = \frac{1}{2}\sum_{i=1}^{d}\beta\,(P_i,P_i) + \frac{1}{2}\sum_{i_1<\cdots<i_M}\beta\,([X_{i_1},\cdots,X_{i_M}]_\lambda,[X_{i_1},\cdots X_{i_M}]_\lambda), \tag{6}$$

where X^i and P_i are elements of (possibly finite dimensional, λ-dependent) vectorspaces V on which antisymmetric M-linear maps $[,\cdots,]_\lambda : V\times\cdots\times V \to V$ are defined, and β a positive definite hermitean form, preferably invariant with respect to some analogue of volume preserving diffeomorphisms (cp. (2)).

With

$$[T_{a_1},\cdots,T_{a_M}]_\lambda = f^a_{a_1\cdots a_M}(\lambda)\,T_a \tag{7}$$

and

$$\beta(T_a,T_b) = \delta^a_b \tag{8}$$

for some (possibly λ-dependent) basis $\{T_a\}_{a=1}^{\dim V}$ of V, i.e.

$$f^a_{a_1\cdots a_M}(\lambda) = \beta(T_a,[T_{a_1},\cdots,T_{a_M}]_\lambda)\,, \tag{9}$$

(6) reads

$$\begin{aligned} H_\lambda = \frac{1}{2}p^*_{ia}p_{ia} + \frac{1}{2}&(f^a_{a_1\cdots a_M}(\lambda))^*\,f^a_{b_1\cdots b_M}(\lambda)\\ &\frac{1}{M!}\,x^*_{i_1a_1}\cdots x^*_{i_Ma_M}\,x_{i_1b_1}\cdots x_{i_Mb_M}\,, \end{aligned} \tag{10}$$

while (1) may be written as

$$\begin{aligned} H = \frac{1}{2}p^*_{i\alpha}p_{i\alpha} + \frac{1}{2}&(g^\alpha_{\alpha_1\cdots\alpha_M})^*\,g^\alpha_{\beta_1\cdots\beta_M}\\ &\frac{1}{M!}\,x^*_{i_1\alpha_1}\cdots x_{i_M\beta_M}\,; \end{aligned} \tag{11}$$

$$g^\alpha_{\alpha_1\cdots\alpha_M} := \int_\Sigma Y^*_\alpha\{Y_{\alpha_1},\cdots,Y_{\alpha_M}\}\,\rho\,d^M\varphi \tag{12}$$

is defined with respect to some orthonormal basis of functions (on Σ) satisfying

$$\begin{aligned} \int Y^*_\alpha\,Y_\beta\,\rho\,d^M\varphi &= \delta^\alpha_\beta\\ \alpha,\beta &= 1\cdots\infty \end{aligned} \tag{13}$$

(even for real x_i, it is often convenient to take a complex basis).
Obvious questions are:

1) Does there exist a 'natural' sequence of finite dimensional vectorspaces V_n with basis $\{T_a^{(n)}\}$ and antisymmetric maps $F_n : V_n \times \cdots \times V_n \to V_n$ such that for each $(M+1)$-tuple $(a\, a_1 \cdots a_M)$

$$\lim_{n\to\infty} f^a_{a_1\cdots a_M}(\lambda_n) \underset{=}{?} g^a_{a_1\cdots a_M} \,. \tag{14}$$

2) For which M do there exist finite dimensional analogues of (2), $K(n)$, leaving $(10)_{\lambda_n}$ invariant, such that, as $n \to \infty$, the full invariance under volume-preserving diffeomorphisms is recovered?

3) What about λ-deformations with infinite dimensional V's ?

Let us look at the case of a M-torus, $\Sigma = T^M$:
Choosing

$$Y_{\vec{m}} = e^{i\,\vec{m}\,\vec{\varphi}},\ \vec{m} = (m_1, \cdots, m_M) \in \mathbb{Z}^M,\ \rho \equiv 1, \tag{15}$$

one gets

$$g^{\vec{m}}_{\vec{m}_1\cdots\vec{m}_M} = i^M(\vec{m}_1, \cdots, \vec{m}_M)\, \delta^{\vec{m}}_{\vec{m}_1+\cdots+\vec{m}_M} \tag{16}$$

where $(\vec{m}_1, \cdots, \vec{m}_M) \in \mathbb{Z}$ denotes the determinant of the corresponding $M \times M$ Matrix (an element of $GL(M,\mathbb{Z})$).

Consider now the following '$*M$-product' (a deformation of the ordinary commutative product of M functions $f_1, \cdots, f_M$ on Σ):

$$(f_1 \cdots f_M)_* := f_1 \cdots f_M + \sum_{m=1}^{\infty} \frac{\left(\frac{(-i)^{M+1}\lambda}{M!}\right)^m}{m!} \cdots \begin{matrix}\epsilon^{r_1 r_1' \cdots r_1^{(M)}} \\ \epsilon^{r_m r_m' \cdots r_m^{(M)}}\end{matrix}$$
$$\frac{\partial^m f_i}{\partial\varphi^{r_1}\cdots\partial\varphi_{r_m}} \cdots \frac{\partial^m f_M}{\partial\varphi^{r_1^{(M)}}\cdots\partial\varphi^{r_m^{(M)}}} \,. \tag{17}$$

One then finds that

$$\begin{aligned}(Y_{\vec{m}_1} \cdots Y_{\vec{m}_M})_* &= \sqrt{\omega}^{\,-(\vec{m}_1,\cdots,\vec{m}_M)}\, Y_{\vec{m}_1+\cdots\vec{m}_M} \\ \sqrt{\omega} &= e^{i\,\frac{\lambda}{M!}} \,.\end{aligned} \tag{18}$$

Defining

$$[f_1, \cdots, f_M]_* := \sum_{\sigma\,\in\,S_M} (\text{sign}\,\sigma)\,(f_{\sigma\,1} \cdots f_{\sigma\,M})_* \tag{19}$$

to simply be the antisymmetrized $*M$-product, one gets

$$[T_{\vec{m}_1}, \cdots, T_{\vec{m}_M}] = \frac{-i}{2\pi\Lambda}\, \sin\left(2\pi\Lambda\,(\vec{m}_1, \cdots, \vec{m}_M)\right) T_{\vec{m}_1+\cdots+\vec{m}_M} \tag{20}$$

$$\text{with } \Lambda := \frac{\lambda}{2\pi M!} \quad \text{and} \quad T_{\vec{m}} := \lambda^{-\frac{1}{M-1}}\, Y_{\vec{m}}.$$

For $M > 1$ arbitrary (but fixed), let V denote the vectorspace (over $\mathbb{C}$) generated by $\{T_{\vec{m}}\}_{\vec{m}\,\in\,\mathbb{Z}^M}$, $\mathbb{M}^\Lambda$ denote $(V, *)$ and $\mathbb{A}^\Lambda$ denote $(V, [\cdots]_*)$.

The hermitean form β (cp. (8),(9)),

$$\beta\,(T_{\vec{m}},T_{\vec{n}}) \;=\; \delta^{\vec{m}}_{\vec{n}}, \quad \beta(c_i\,X_i,d_j\,X_j) \;=\; c^*_i\,d_j\;\beta(X_i,X_j),$$

will have the important property ('invariance') that (for $X_i = x_{i\vec{m}}T_{\vec{m}}$ with $x^*_{i\vec{m}} = x_{i-\vec{m}}$)

$$\beta\;(X,[X_{i_1},\cdots X_{i_M}]) \;=\; -\beta\left(X_{i_r},[X_{i_1},\cdots,X_{i_{r-1}},X,X_{i_{r+1}},\cdots,X_{i_M}]\right),$$

as

$$\beta\;(T_{\vec{m}},[T_{\vec{m}_1},\cdots,T_{\vec{m}_M}]) \;=\; \frac{-\,i}{2\pi\Lambda}\,\delta^{\vec{m}}_{\vec{m}_1,+\cdots+\vec{m}_M}\;\sin\left(2\pi\Lambda(\vec{m}_1,\cdots,\vec{m}_M)\right).$$

For rational $\Lambda = \frac{\widetilde{N}}{N}$ (assuming N and $\widetilde{N} < N$ having no common divisor > 1) both $\mathbb{A}^\Lambda$ and $\mathbb{M}^\Lambda$ may be divided by an ideal of finite codimension, namely (using the periodicity of the structure-constants) the vectorspace I generated by all elements of the form $T_{\vec{m}} - T_{\vec{m}+N\;(\text{anything})}$. One thus arrives at considering (for arbitrary odd N)

$$V^{(N)} \;:=\; \left\langle T_{\vec{m}}|m_r \;=\; -\,\frac{N-1}{2}\,,\cdots,\;+\;\frac{N-1}{2}\right\rangle_{\mathbb{C}} \qquad r \;=\; 1\cdots M \tag{21}$$

with a $*_M$ product on $V^{(N)}$ defined just as in (18):

$$\begin{aligned}(T_{\vec{m}_1}\cdots T_{\vec{m}_M})_* \;&:=\; \frac{-\,i\,N}{2\pi\widetilde{N}M!}\;\omega^{-\frac{1}{2}\,(\vec{m}_1,\cdots,\vec{m}_M)}\;T_{\vec{m}_1+\cdots+\vec{m}_M\;(\text{mod }\;N)}\\ \omega \;&=\; e^{4\pi\,i\,\frac{\widetilde{N}}{N}}\;,\end{aligned} \tag{22}$$

and a corresponding alternating product,

$$\begin{aligned}&[T_{\vec{m}_1},\cdots,T_{\vec{m}_M}]_* \;=\; \frac{-\,i\,N}{2\pi\widetilde{N}}\sin\,(2\pi\;\frac{\widetilde{N}}{N}\;(\vec{m}_1,\cdots,\vec{m}_M))\;T_{\vec{m}_1+\cdots+\vec{m}_M\;\;(\text{mod }\;N)}\\ &\vec{m}_r \;\in\; (\mathbb{Z}_N)^M\;.\end{aligned} \tag{23}$$

The 'structure constants' of the alternating finite dimensional M-algebras

$$\begin{aligned}\mathbb{A}_N \;&:=\; (V^{(N)},[,\cdots,]_*),\\ f^{(N)\,\vec{m}}_{\vec{m}_1\cdots\vec{m}_M} \;&:=\; \frac{-\,i\,N}{2\pi\widetilde{N}}\sin\left(2\pi\;\frac{\widetilde{N}}{N}\;\Big(\vec{m}_1,\cdots,\vec{m}_M\Big)\right)\;\cdot\;\delta^{\vec{m}}_{\vec{m}_1+\cdots+\vec{m}_M\;\;(\text{mod }\;N)}\end{aligned} \tag{24}$$

satisfy (14) (up to an N and $\mathbb{Z}^M_N$-independent rescaling of the generators, resp. factors of i, which anyway drop out in (10) and (11); $n \;=\; N^M$, $f^{(N)} \;\stackrel{\wedge}{=}\; f(\lambda_n)$, $\vec{m} \;\in\; \mathbb{Z}^M_N$ $V^{(N)} \;=\; V_{n=N^3}$, and $\lim\limits_{N\to\infty}\;V^{(N)} = V$).

$$\begin{aligned}H_N \;\;&=\;\; \frac{1}{2}\,p_{i-\vec{m}}\;p_{i\vec{m}}\\ &+\;\; \frac{1}{2}\;\frac{N^2}{4\pi^2\;\widetilde{N}^2}\sin\left(2\pi\frac{\widetilde{N}}{N}\;\Big(\vec{m}_1\cdots\vec{m}_M\Big)\right)\;\cdot\;\sin\left(2\pi\;\frac{\widetilde{N}}{N}\;\Big(\vec{n}_1,\cdots\vec{n}_M\Big)\right)\\ &\;\;\;\;\frac{1}{M!}\;\cdot\;x_{i_1-\vec{m}_1}\cdots x_{i_M-\vec{m}_M}\;x_{i_1\vec{n}_1}\cdots x_{i_M\vec{n}_M}\;\delta^{\vec{m}_1+\cdots+\vec{m}_M}_{\vec{n}_1+\cdots+\vec{n}_M\;\;(\text{mod }\;N)}\end{aligned} \tag{25}$$

could therefore be considered as a finite-dimensional analogue of (1).

3. Multidimensional Commutation Relations

Before turning to questions of symmetry, let me discuss in a little more detail the $*M$-algebras $\mathbb{M}^\Lambda$, with defining relations (cp. (18); note the slight change of notation/normalisation)

$$(T_{\vec{m}_1}\cdots T_{\vec{m}_M})_* \;=\; \omega^{-\frac{1}{2}\,(\vec{m}_1,\cdots,\vec{m}_M)}\, T_{\vec{m}_1+\cdots+\vec{m}_M}\ (*)\ ,$$

and as vectorspaces generated by basis-elements $T_{\vec{m}}$, $\vec{m}\in S$ (where $S=\mathbb{Z}^M$, $S=(\mathbb{Z}_N)^M$, or any combination thereof – in the M-brane context, depending on whether $\Sigma = T^M$, resp. a fully, or partially, discretized M-torus).

First of all note, that for any M elements, $A_1,\cdots A_M\in V$, any even permutation $\sigma\in S_M$ (the symmetric group in M objects), and any choice of S (even $\mathbb{R}^M$),

$$(A_1\cdots A_M)_* \;=\; (A_{\sigma(1)}\cdots A_{\sigma(M)}) \quad (\text{sign}\,\sigma \;=\; +)\ , \tag{26}$$

and that $E := T_{\vec{0}}$ acts as a 'unity' in the sense that if one of the A_r is equal to $T_{\vec{0}}$, the $*M$-product becomes commutative (i.e. independent of the order of its M entries).

Using E, one may identify $T_{(m=\pm|m|,0,\cdots,0)}$ with the $|m|$-th power of $E_{\pm1} := T_{(\pm1,0,\cdots,0)}$,

$$T_{(m,0,\cdots,0)} \;=\; \underset{\substack{\uparrow\\ |m|\ \text{brackets}}}{((((}E\cdots EE_{\pm1})_*\cdots EE_{\pm1})_*\cdots)_*\cdots EE_{\pm1})_*\ , \tag{27}$$

so that one may wonder whether $\mathbb{M}^\Lambda$ can be thought of as being generated by

$$E \;=\; T_{\vec{0}},\ E_{\pm1} \;=\; T_{(\pm1\ 0\cdots0)},\cdots,\ E_{\pm M} \;=\; T_{(0\cdots0\ \pm1)}\ .$$

This is indeed the case: Let $\mathbb{F}^M$ be the free (non associative) M-algebra generated by $2M+1$ elements $E, E_{\pm1},\cdots,E_{\pm M}$; define arbitrary powers $(E_r)^m$ of the generating elements as in (27) (from now on $E_{-r}^{|m|} =: E_r^{-|m|}$, a notation that will be justified via (29)), and let

$$E_{\vec{m}} \;:=\; E_1^{m_1}\, E_2^{m_2}\ \cdots\ E_M^{m_M}\ . \tag{28}$$

Divide $\mathbb{F}^M$ by the ideal generated by elements

$$E_{\vec{m}'}\, E_{\vec{m}''}\cdots E_{\vec{m}^{(M)}} \;-\; \omega^{\gamma(\vec{m}',\vec{m}'',\cdots,\vec{m}^{(M)})}\ \cdot\ E_{\vec{m}'+\cdots+\vec{m}^{(M)}} \tag{29}$$

where $\omega = e^{4\pi i\Lambda}$ and

$$\begin{aligned} 2\gamma(\vec{m}',\cdots,\vec{m}^{(M)}) \;&:=\; (m_1\cdot m_2\cdot\ \cdots\ \cdot m_M) \;-\; (\vec{m}',\vec{m}'',\cdots,\vec{m}^{(M)}) \\ &\quad -\ \sum_{r=1}^{M}\Big(\prod_{s=1}^{M} m_s^{(r)}\Big) \\ (\vec{m} \;:=\; \vec{m}' \,+\, \vec{m}'' \;&+\; \cdots \,+\, \vec{m}^{(M)})\ . \end{aligned} \tag{30}$$

This quotient then is isomorphic to $\mathbb{M}^\Lambda$, as can be seen by defining

$$T_{\vec{m}} \;:=\; \omega^{\frac{1}{2}\,m_1m_2\cdots m_M}\, E_1^{m_1}\, E_2^{m_2}\cdots E_M^{m_M}\ , \tag{31}$$

which (due to (29) being zero in $\mathbb{F}^{\Lambda}/I$) satisfies (18) (with Y standing for T).

Note that

$$E_2^{m_2}\, E_1^{m_1}\, E_3^{m_3}\, \cdots\, E_M^{m_M} \;=\; \omega^{m_1 m_2 \cdots m_M}\, \cdot\, E_1^{m_1}\, E_2^{m_2} \cdots E_M^{m_M}, \tag{32}$$

in particular:

$$E_2\, E_1\, E_3\, \cdots\, E_M \;=\; \omega\, E_1\, E_2\, \cdots\, E_M \tag{33}$$

(while any even permutation does not alter the product, cp. (26)).

In order to get a feeling for (29)/(30) it may be instructive to consider $M=3$: due to (29),

$$\begin{array}{rl} & (E_1^{n_1}\, E_2^{n_2}\, E_3^{n_3})(E_1^{l_1}\, E_2^{l_2}\, E_3^{l_3})(E_1^{k_1}\, E_2^{k_2}\, E_3^{k_3}) \\ = & E_1^{n_1+l_1+k_1}\, E_2^{n_2+l_2+k_2}\, E_3^{n_3+l_3+k_3} \\ & \cdot\, \omega^{n_1 l_3 k_2 + n_2 l_1 k_3 + n_3 l_2 k_1} \\ & \cdot\, \sqrt{\omega}^{\,n_1(l_2 l_3 + k_2 k_3) + n_2(l_1 l_3 + k_1 k_3) + n_3(l_1 l_2 + k_1 k_2)} \\ & \cdot\, \sqrt{\omega}^{\,n_1 n_2(l_3+k_3) + n_1 n_3(l_2+k_2) + n_2 n_3(l_1+k_1)} \qquad . \end{array} \tag{34}$$

The general rule (30) can hence be stated as follows:

Consider all possible triples (resp. M-tuples) containing powers of each of the $E_r (r = 1 \cdots M)$ exactly once. If the 'contraction' picks out exactly one factor from each of the 3 (resp. M) factors in (34) it does <u>not</u> contribute if they are already in the correct order, modulo even permutations (cp. 26), (like $E_1^{n_1}\, E_2^{l_2}\, E_3^{k_3}$, or $E_2^{n_2}\, E_3^{l_3}\, E_2^{k_1}$), while they contribute a factor $\omega^{(\text{product of the } E-\text{powers})}$, when they are <u>not</u> in the correct (modulo even permutation) order (like $E_2^{n_2}\, E_1^{l_1}\, E_3^{k_3}$). Contractions entirely within one of the factors don't contribute, while mixed contractions (involving at least 2, but not all, of the factors in (34)), all contribute a factor $\sqrt{\omega}^{(\text{product of the } E-\text{powers})}$, irrespective of their order.

Due to (32), all 'monomials' are proportional to one of the elements $E_{\vec{m}}$ (cp. (28)) – which therefore form a basis (with the convention $E_{\vec{0}} \equiv E$). Note that $2\pi M!\,\Lambda = \lambda \to 0$ is a 'classical limit' (resp. $\lambda \neq 0$ a 'quantisation' of the classical Nambu-structure) as, formally,

$$[\ln\, E_1, \ln\, E_2, \cdots, \ln\, E_M] \;=\; i\,\lambda\, E\;. \tag{35}$$

Having obtained this relation, one could of course start with objects $\ln E_r =: J_r$, $[J_1, J_2, \cdots, J_M] = i\,\lambda\, E$, and derive generalized 'Hausdorff-formulae' for products involving the $e^{i\, m_r\, J_r}$.

Of course, (35) cannot be true in any M-algebra containing only finite linear combinations of the basis-elements $E_{\vec{m}}$, as $T_{\vec{0}} = E$ never appears on the r.h.s. of (20); this is similar to the fact that the canonical commutation relations of ordinary quantum mechanics, $[q, p] = i\,\hbar\,\mathbf{1}$, cannot hold for trace-class operators. (35) may be justified by formally expanding $\ln\, E_r \;=\; -\sum\limits_{n_r=1}^{\infty}\ \sum\limits_{k_r=0}^{n_r} \binom{n_r}{k_r} \dfrac{(-)^{k_r}}{n_r}\, E_r^k$, using

$$[E_1^{k_1}, E_2^{k_2}, \cdots, E_M^{k_M}] \;=\; \frac{M!}{2}\,(1 - \omega^{k_1 \cdots k_M})\, E_1^{k_1} \cdots E_M^{k_M}$$

and then resumming recursively, after the first step obtaining

$$\frac{M!}{2}\ln E_1\cdots\ln E_M \;-\; \frac{M!}{2}\sum_{\substack{n_r,k_r\\ r>1}}{}'\cdots\ln(E_1\omega^{k_2\cdots k_M})E_2^{k_2}\cdots E_M^{k_M} \;=\; \frac{M!}{2}(\ln\omega)\,\cdot\,E\;, \tag{36}$$

as formally,

$$\sum_{n_r=1}^{\infty}\sum_{k_r=1}^{n_r}\binom{n_r}{k_r}\frac{(-)^{k_r}}{n_r}\,k_r\,E_r^k \;=\; E_r\,\cdot\,\sum_{n'=0}^{\infty}(E-E_r)^{n'} \;=\; E\;.$$

4. Breakdown of Conventional Symmetries

Let us now discuss the question, whether theories like (5) or (6) can have symmetries reminiscent of volume preserving diffeomorphisms; in particular whether the generators (2) may be 'translated' to finite dimensional analogues. * For simplicity, consider again $\Sigma = T^M$.

As $f^r = \partial_s\omega^{rs} = \epsilon^{rsr_1\cdots r_{M-2}}\partial_s\omega_{r_1\cdots r_{M-2}}$ for non-constant (divergence-free) vector-fields on T^M, (2) may be written in the form

$$K_{r_1\cdots r_{M-2}} \;=\; \int d^M\varphi\;\omega_{r_1\cdots r_{M-2}}\,\{p_i,x^i,\varphi^{r_1},\cdots,\varphi^{r_{M-2}}\}\;, \tag{37}$$

resp., in Fourier-components,

$$K^{\vec{l}}_{r_1\cdots r_{M-2}} \;=\; \sum_{\substack{\vec{m},\vec{n}\\ \in\mathbb{Z}^M}}\delta^{\vec{l}}_{\vec{m}+\vec{n}}\;p_{i\vec{m}}\,x_{i\vec{n}}\,(\vec{m},\vec{n},\vec{e}_{r_1},\cdots,\vec{e}_{r_{M-2}}) \tag{38}$$

(where $\vec{e}_r$ denotes the unit vector in the r-direction).

Suppose the deformed theory was invariant under transformations that are still generated in a conventional way by phase-space functions of the form

$$K^{\vec{l}} \;=\; \sum_{\vec{m},\vec{n}\,\in S} p_{i\vec{m}}x_{i\vec{n}}\;\delta^{\vec{l}}_{\vec{m}+\vec{n}}\;c_{\vec{m}\vec{n}}\;. \tag{39}$$

Using $[x_{i\vec{m}},p_{j\vec{n}}] = \delta_{ij}\delta^{-\vec{n}}_{\vec{m}}$, while leaving open whether $S=\mathbb{Z}^M$ or $S=(\mathbb{Z}_N)^M$ as well as (independently) whether δ is defined $\bmod N$, or not, one has

$$[K^{\vec{l}},\tilde{K}^{\vec{l}'}] \;=\; \sum_{\substack{\vec{m}_1\vec{n}\\ \in S}} p_{i\vec{m}}x_{i\vec{n}}\;\delta^{\vec{l}+\vec{l}'}_{\vec{m}+\vec{n}}\;\approx{c}_{\vec{m}\vec{n}}$$

with (40)

$$\approx{c}_{\vec{m}\vec{n}} \;=\; \sum_{\vec{k}\,\in S}\left(\delta^{\vec{l}-\vec{m}}_{\vec{k}}\;\delta^{\vec{l}'-\vec{n}}_{-\vec{k}}\;c_{\vec{m}\vec{k}}\;\tilde{c}_{-\vec{k}\vec{n}} \;-\; \begin{pmatrix}\vec{l}\leftrightarrow\vec{l}'\\ c\leftrightarrow\tilde{c}\end{pmatrix}\right)\;,$$

*For $M=2$, this question was already considered in [4] and answered positively.

while $\dot{K}^{\vec{l}} = 0$ would require $c_{\vec{m}\vec{n}} = -- \ c_{\vec{n}\vec{m}}$ and

$$\sin\left(2\pi\Lambda(\vec{a}_1, \cdots, \vec{a}_M)\right) \ \sin\left(2\pi\Lambda(\vec{a}_1 + \cdots + \vec{a}_M, \vec{a}_2{}', \cdots, \vec{a}_M{}')\right) \cdot c_{\vec{a}_1 + \cdots \vec{a}_1{}' + \cdots \vec{a}_M{}', \vec{a}_1{}'} \ \cdot \ x_{i_1\vec{a}_1} \, x_{i_1\vec{a}_1{}'} \ \cdots \ x_{i_M\vec{a}_M} \, x_{i_M\vec{a}_M{}'} \ = \ 0 \tag{41}$$

(where for (41) consistency of the δ-functions used in (39) and $(25)_\Lambda$ with the index set S was assumed).

The effect of the $x_{i\vec{m}}$-factors in (41) is to make the product $\sin\cdot\sin\cdot c$, symmetric under any interchange $\vec{a}_r \leftrightarrow \vec{a}_r{}'$, as well as any simultaneous interchange $\vec{a}_r \leftrightarrow \vec{a}_s$, $\vec{a}_r{}' \leftrightarrow \vec{a}_s{}'$. Choosing, e.g., $\vec{a}_r{}' = \vec{a}_r (r = 1 \cdots M)$, with $\sin(2\pi\Lambda(\vec{a}_1 \cdots \vec{a}_M)) \neq 0$, (41) requires that

$$\sum_{\sigma \in S_M} c_{\vec{a}_{\sigma 1} + 2(\vec{a}_{\sigma 2} + \cdots + \vec{a}_{\sigma M}), \vec{a}_{\sigma 1}} \ = \ 0 \ . \tag{42}$$

This condition is insensitive to any alteration of the deformation: replacing the sine-function in (41) (resp. $(25)_\Lambda, \cdots$) by any other function of the determinant will still result in (42) as a necessary condition. Apart from $M = 2$ ($c_{\vec{a}_1 + 2\vec{a}_2, \vec{a}_1} + c_{\vec{a}_2 + 2\vec{a}_1, \vec{a}_2} = 0$ is trivially satisfied by any odd function) (42) is not satisfied by

$$c_{\vec{m}\vec{n}} \ = \ \sin(2\pi\Lambda(\vec{m}, \vec{n}, \vec{k}_1, \cdots, \vec{k}_{M-2})) \ , \tag{43}$$

- – nor would (40) be a linear combination of the generators (39), for such a $c_{\vec{m}\vec{n}}$; for $M = 3$, e.g., one would obtain

$$\begin{aligned}
&\approx\!\!\!\!\!\!c_{\vec{m}\vec{n}}(\vec{l}\,\vec{l}'; \vec{k}\,\vec{k}\,') \\
= \ & \sin\Big(2\pi\Lambda\left(\vec{l}, \, \vec{l}', \frac{\vec{k}+\vec{k}\,'}{2}\right)\Big) \\
& \quad \cdot \ \sin\left(2\pi\Lambda\left(\Big(\vec{m}, \vec{n}, \ \frac{\vec{k}+\vec{k}\,'}{2}\Big) \ + \ \Big(\vec{m}-\vec{n}, \frac{\vec{l}-\vec{l}'}{2}, \ \frac{\vec{k}-\vec{k}\,'}{2}\Big)\right)\right) \\
- \ & \sin\Big(2\pi\Lambda\left(\vec{l}, \vec{l}', \ \frac{\vec{k}-\vec{k}\,'}{2}\right) \\
& \quad \cdot \sin\Big(2\pi\Lambda\left(\vec{m}, \vec{n}, \ \frac{\vec{k}-\vec{k}\,'}{2}\right) \ + \ \Big(\vec{m}-\vec{n}, \ \frac{\vec{l}-\vec{l}'}{2}, \ \frac{\vec{k}+\vec{k}\,'}{2}\Big)\Big)
\end{aligned} \tag{44}$$

- – which means that the algebra closes only for $\vec{k}\,' = \vec{k}$ (for $\Lambda = \frac{1}{N}$ this would give N^3 closed Lie algebras, each of dimension N^3; in fact, each consisting of N copies of $gl(N)$). – In any case, if $c_{\vec{m}\vec{n}}$ was a function of $(\vec{m}_1\vec{n}_1\vec{k}_1, \cdots, \vec{k}_{M-2})$, one could let $\vec{a}_2, \vec{a}_3, \cdots \vec{a}_M$ differ only in the ('irrelevant') $\vec{k}_1, \cdots \vec{k}_{M-2}$ directions and obtain

$$f\ (((2M-2)\vec{a}_2, \vec{a}_1, \cdots)) \ + \ (M-1)\, f\,((2\vec{a}_1, \vec{a}_2, \cdots)) \ = \ 0 \ , \tag{45}$$

which eliminates all $c_{\vec{m}\vec{n}}$ that are non-linear functions of the determinant.

Interestingly, $c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{ something})_{\text{if } M>2}$ is suggested by yet another consideration: replacing $\{p_i, x_i, \varphi^3, \cdots, \varphi^M\}$ (cp. (37); for notational simplicity taking $r_1 = 3, \cdots, r_{M-2} = M$) by

$$[P_i, X_i, \ln E_3, \cdots, \ln E_M] \ , \tag{46}$$

(with $P_i = p_{i\vec{m}} T_{\vec{m}}, X_i = x_{i\vec{m}} T_{\vec{m}}$) formally expanding the logarithms in a power series, using (20), and then (formally) summing again, one obtains something proportional to

$$p_{i\vec{m}}\, x_{i\vec{n}}\, T_{\vec{m}+\vec{n}} \;\cdot\; (m_1\, n_2 - m_2\, n_1)\ . \tag{47}$$

$$\begin{aligned}
& [P_i, X_i, \ln E_3, \cdots, \ln E_M] \\
= \;& p_{i\vec{m}}\ x_{i\vec{n}}\ (-)^{M-2} \sum_{n_3=1}^{\infty} \sum_{k_3=0}^{n_3} \cdots \sum_{n_M=1}^{\infty} \sum_{k_M=0}^{n_M} \binom{n_3}{k_3} \cdots \binom{n_M}{k_M} \frac{(-)^{k_3+\cdots+k_M}}{n_3 \cdots n_M} \\
& \quad \cdot [T_{\vec{m}},\ T_{\vec{n}},\ E_3^{k_3}, \cdots, E_M^{k_M}] \\
\sim \;& \sum \cdots \sin\left(2\pi\Lambda\left(\vec{m}, \vec{n}, k_3\, \vec{e}_3, \cdots, k_M\, \vec{e}_M\right)\right) \;\cdot\; T_{\vec{m}+\vec{n}+\vec{k}} \\
\sim \;& \sum \cdots \left(\sqrt{\omega}^{\,k_3 \cdots k_M\ z} - \sqrt{\omega}^{\,-k_3 \cdots k_M\ z} \right)\ (\sqrt{\omega})^{\prod_{r=1}^{M} (m_r + n_r + k_r)}\ \cdot \\
& \quad \cdot\ E_1^{m_1+n_1}\ E_2^{m_2+n_2}\ E_3^{m_3+n_3+k_3}\ \cdots\ E_M^{m_M+n_M+k_M} \\
\sim \;& \sum{}' \cdots \Big(\ln\Big(\sqrt{\omega}^{\,k_4 \cdots k_M\ z + \prod_{r\neq 3}(m_r+n_r+k_r)}\ E_3 \Big) \\
& \quad -\ \ln\Big(\sqrt{\omega}^{\,-k_4 \cdots k_M\ z + \prod_{r\neq 3}(\cdots)}\ E_3 \Big) \Big) \;\cdot\; \sqrt{\omega}^{\,(m_3+n_3)\cdot \prod_{r\neq 3}(\cdots)} \\
& \quad \cdot\ E_1^{m_1+n_1}\ E_2^{m_2+n_2}\ E_3^{m_3+n_3}\ E_4^{m_4+n_4+k_4}\ \cdots\ E_M^{m_M+n_M+k_M} \\
& \begin{pmatrix} z\ :=\ (\vec{m}, \vec{n}, \vec{e}_3, \cdots, \vec{e}_M)\ =\ m_1\, n_2 - m_2\, n_1 \\ \vec{k}\ =\ (0,\ 0,\ k_3, \cdots, k_M) \end{pmatrix} \\
= \;& (\ln \omega)\ p_{i\vec{m}}\ x_{i\vec{n}}\ z\ (\vec{m}, \vec{n})\ \sqrt{\omega}^{\,\prod_1^M (m_r+n_r)}\ E_1^{m_1+n_1} \cdots E_M^{m_M+n_M} \\
= \;& (m_1\ n_2 - m_2\ n_1)\ p_{i\vec{m}}\ x_{i\vec{n}}\ (\ln \omega)\ \cdot\ T_{\vec{m}+\vec{n}} \\
& \text{where (for } r > 3) - \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{(-)^k}{n}\ k\ \cdot\ E_r^k\ \cdot\ (\omega^{\cdots})^k\ =\ E \text{ was used.}
\end{aligned}$$

However,

$$c_{\vec{m}\vec{n}}\ =\ (\vec{m}, \vec{n},\ \text{anything}) \tag{48}$$

does <u>not</u> satisfy (41). Moreover, even if one considers more general deformations of the Hamiltonian, i.e. replacing the sine-function in (41) by an arbitrary odd (power-series) function f of the determinant, the corresponding condition,

$$\begin{aligned} f\,(\vec{a}_1, \cdots, \vec{a}_M)\ f\,(\vec{a}_1 + \cdots + \vec{a}_M,\ \vec{a}'_2, \cdots, \vec{a}'_M)\ \cdot (\vec{e}, \vec{a}'_1, \cdots)\ &=\ 0 \\ +\ (M\ \cdot\ 2^M - 1)\ \text{permutations}\ , & \end{aligned} \tag{49}$$

$\vec{e} = \sum_{r=1}^{M} (\vec{a}_r + \vec{a}'_r)$, can never be satisfied by any non-linear function f – as on can see, e.g., by choosing $\vec{a}'_r = \mu_r \vec{a}_r$. Supposing $f(x) = \alpha x + \beta x^{2n+1} = \cdots$, and denoting $(\vec{a}_1, \cdots, \vec{a}_M)$ by z, $\prod_{r=1}^{M} \mu_r$ by μ, the terms $\mu_1, \alpha\, z\, \beta\ (\mu\, z)^{2n+1}$, e.g., (occurring only twice, with the same sign) could never cancel.

The preceding arguments possibly suffice to prove that, independent of the above dynamical context, the Lie algebra of volume-preserving diffeomorphisms of $T^{M>2}$ does not possess any non-trivial deformations.*

5. Rigidity of Canonical Nambu-Poisson Manifolds

For the multilinear antisymmetric map (4), and $2M-1$ arbitrary functions $f_1, \cdots, f_{2M-1}$, one has (cp. [5]):

$$\begin{aligned} &\{\{f_M, f_1, \cdots, f_{M-1}\},\ f_{M+1}, \cdots,\ f_{2M-1}\} \\ &\quad + \{f_M, \{f_{M+1}, f_1, \cdots, f_{M-1}\}, f_{M+2}, \cdots, f_{2M-1}\} \\ &\quad + \cdots + \{f_M, \cdots f_{2M-2}, \{f_{2M-1}, f_1, \cdots, f_{M-1}\}\} \\ &\quad = \{\{f_M, \cdots, f_{2M-1}\}, f_1, \cdots, f_{M-1}\}\ . \end{aligned} \tag{50}$$

Takhtajan [5], stressing its relevance for time-evolution in Nambu-mechanics [1], named (50) 'Fundamental Identity (FI)', and defined a 'Nambu-Poisson-manifold of order M' to be a manifold X together with a multilinear antisymmetric map $\{\cdots\}$ satisfying (50) and the Leibniz-rule

$$\{f_1\tilde{f}_1, f_2, \cdots, f_M\} = f_1\{\tilde{f}_1, f_2, \cdots, f_M\} + \{f_1, \cdots, f_M\}\,\tilde{f}_1 \tag{51}$$

for functions $f_r : X \to \mathbb{R}$ (or $\mathbb{C}$).

Without (51), i.e. just demanding (50) for an antisymmetric M linear map: $V \times \cdots \times V \to V$, V some vectorspace, Takhtajan defines a 'Nambu-Lie-gebra' [5], – also called 'Fillipov [6] Lie algebra' [7]). I would like to point out various other identities satisfied by canonical Nambu-Poisson brackets (4), and show that all of them – including (50)! – do <u>not</u> allow deformations (of certain natural type), if $M > 2$.

At least from a non-dynamical point of view, all identities involving Nambu-brackets obtained from antisymmetrizing the product of two determinants formed from $2M$ M-vectors,

$$(\vec{a}_1 \cdots \vec{a}_M)(\vec{a}_{M+1} \cdots \vec{a}_{2M}) \tag{52}$$

with respect to $M+1$ of the $\vec{a}_\alpha (\alpha = 1 \cdots 2M)$ should be treated on an equal footing. For $M = 3$, e.g., one has – apart from

$$\begin{aligned} &(\vec{a}\,\vec{b}\,\vec{c}_1)(\vec{c}_2\,\vec{c}_3\,\vec{c}_4) - (\vec{a}\,\vec{b}\,\vec{c}_2)(\vec{c}_3\,\vec{c}_4\,\vec{c}_1) \\ &+ (\vec{a}\,\vec{b}\,\vec{c}_3)(\vec{c}_4\,\vec{c}_1\,\vec{c}_2) - (\vec{a}\,\vec{b}\,\vec{c}_4)(\vec{c}_1\,\vec{c}_2\,\vec{c}_3) = 0\ , \end{aligned} \tag{53}$$

which gives rise to $(50)_{M=3}$ for functions $f \in T^3$ – also

$$(a\,\vec{c}_{[1}\,\vec{c}_2)(\vec{c}_3\,\vec{c}_{4]}\,\vec{b}) = 0\ , \tag{54}$$

*M. Bordemann has informed me that apparently an even more general statement of this nature has recently been proven in [19].

yielding the following 6-term identity $(\mathrm{FI})_6$ (which can of course also be proven by using just the definition (4), $\{f,g,h\} = \epsilon_{\alpha\beta\gamma}\ \partial_\alpha\ f\ \partial_\beta\ g\ \partial_\gamma\ h$, rather than (54); i.e. not necessarily specifying the manifold X):

$$\{\{f,\ f_{[1},\ f_2\}\ f_3,\ f_{4]}\}\ =\ 0 \tag{55}$$

as well as the 4-term identity $(\widetilde{\mathrm{FI}})$,

$$\begin{aligned} &\{\{f,f_1,f_2\},\ g,f_3\} \\ +\ &\{\{f,f_2,f_3\},\ g,f_1\} \\ +\ &\{\{f,f_3,f_1\},\ g,f_2\}\ =\ -\ \{f,g,\ \{f_1,f_2,f_3\}\} \end{aligned} \tag{56}$$

- – each of which is independent of $(50)_{M=3}$ (while any 2 of the 3 identities yield the 3rd).

Naively, one would think that (56) should follow from $(50)_3$ alone, as (54) follows from (53) (perhaps one should note that for general M, a theorem concerning vector invariants [8] states, that any (!) vector-bracket identity is an algebraic consequence of

$$(\vec{a}_{[1}\,\vec{a}_2\ \cdots\ \vec{a}_M)\ (\vec{a}_{M+1]}\ \cdots\ \vec{a}_{2M})\ =\ 0\ ;$$

however, in the proof of (56) via vector-bracket identities, one in particular needs (54) for the special case $\vec{a} = \vec{b}$ – which cannot be stated as an identity between functions on X .) Curiously (with respect to a statistical approach to M-branes), vector-bracket identities ('Basis Exchange Properties' [9]) also play an important role in combinatorical geometry.

¿From an aesthetic point of view, the most natural quadratic identity for (4) is

$$\sum_{\sigma\,\in\,S_{2M-1}} (\mathrm{sign}\ \sigma)\{\{f_{\sigma 1},\cdots,f_{\sigma M}\}\ f_{\sigma M+1},\cdots,f_{\sigma 2M-1}\}\ =\ 0\ . \tag{57}$$

For $M = 3$, e.g., one could see this to be a consequence of $(50)_3$ and (56) by grouping the 10 distinct terms in (57) according to whether $\{f_{\sigma 1}, f_{\sigma 2}, f_{\sigma 3}\}$ contains both f_4 and f_5 (3 terms, 'type A'), only one of them (3 'B-terms' and 3 'C-terms') or none of them (1 term, 'type D'); for the B (resp. C)-terms one can use (56) while (50) for the A-terms, to get $\pm\ \{f_4, f_5, \{f_1\, f_2\, f_3\}\}$ for each of the 4 types, and for the B and C-terms with a sign opposite to the one obtained from the D (and A) term(s). (57) (taken without the derivation-requirement) is a beautiful generalisation of Lie-algebras ($M = 2$), and has recently started to attract the attention of mathematicians – mostly under the name of $(M-1)$-ary Lie algebras [10 - 13]. *

Unfortunately, all identities (50), (55)–(57), can be shown to be rigid, in the following sense: assuming that

$$[T_{\vec{m}_1},\cdots,T_{\vec{m}_M}]_\lambda\ =\ g_\lambda\left((\vec{m}_1,\cdots,\vec{m}_M)\right)T_{\vec{m}_1+\cdots+\vec{m}_M} \tag{58}$$

with $g_\lambda(x)$ a smooth odd function proportional to $x + \lambda^n\, c\, x^n$ as $\lambda \to 0$ $(n > 1)$ any of the above identities will require the constant c to be equal to zero (I have proved this

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* I would like to thank W. Soergel for mentioning refs. [10]/[11] to me and J.L. Loday for sending me a copy of [10] and [12]; also, I would like to express my gratitude to R. Chatterjee and L. Takhtajan for sending me their papers on Nambu Mechanics (cp. [5]).

only for $M = 3$, and in the case of (57) – the a priori most promising case – for general $M > 2$).

Concerning

$$\begin{aligned} & g_\lambda\left((\vec{a},\vec{b},\vec{c}_1)\right)\ g_\lambda\left((\vec{a}+\vec{b}+\vec{c}_1,\vec{c}_2,\vec{c}_3)\right) \\ + \quad & g_\lambda\left((\vec{a},\vec{b},\vec{c}_2)\right)\ g_\lambda\left((\vec{a}+\vec{b}+\vec{c}_2,\vec{c}_3,\vec{c}_1)\right) \\ + \quad & g_\lambda\left((\vec{a},\vec{b},\vec{c}_3)\right)\ g_\lambda\left((\vec{a}+\vec{b}+\vec{c}_3,\vec{c}_1,\vec{c}_2)\right) \\ \stackrel{!}{=} \quad & g_\lambda\left((\vec{c}_1,\vec{c}_2,\vec{c}_3)\right)\ g_\lambda\left((\vec{c}_1+\vec{c}_2+\vec{c}_3,\vec{a},\vec{b})\right)\ , \end{aligned} \tag{59}$$

i.e. the deformation of $(50)_{M=3}$, one could assume $z := (\vec{c}_1,\vec{c}_2,\vec{c}_3) \neq 0$, $\vec{a} = \sum_1^3 \alpha_r \vec{c}_r$, $\vec{b} = \sum_1^3 \beta_r \vec{c}_r$, such that $g(y) := \bar{g}_\lambda(y) := g_\lambda(zy)$ must satisfy

$$\begin{aligned} & g\,(\alpha_2\,\beta_3 - \alpha_3\,\beta_2)\ \cdot\ g\,(1+\alpha_1+\beta_1) \\ & \quad + \text{cyclic permutations} \\ & \quad = \ g\,(1)\ \cdot\ g\,(\alpha_2\,\beta_3 - \alpha_3\,\beta_2\ +\ \text{cycl.}) \end{aligned} \tag{60}$$

for all α_r, β_r; which is clearly impossible for any nonlinear g of the required form, (e.g., as in next to lowest order in λ the terms $\alpha_1(\alpha_2\,\beta_3)^{n>1}$ appear only once).

Similarly, the deformation of (56) is impossible due to the analogous requirement

$$\begin{aligned} & g\,(\alpha_3)\ g\ (\beta_2 - \beta_1\ +\ (\alpha_1\,\beta_2\ -\ \alpha_2\,\beta_1))\ +\ \text{cycl.} \\ & \quad \stackrel{!}{=}\ -g\,(1)\ g\ ((\alpha_1\,\beta_2\ -\ \alpha_2\,\beta_1)\ +\ \text{cycl.})\ . \end{aligned} \tag{61}$$

Finally, concerning possible deformations of (57), let $(\vec{a}_1,\cdots,\vec{a}_M) \neq 0$, and

$$\vec{a}_{M+\bar{r}}\ =\ \sum_{s=1}^{M} \alpha_s^{(\bar{r})}\ \vec{a}_s\ (\bar{r} = 1,\cdots,M-1);$$

$$\text{then}\ g\ (1+\alpha_1^{(1)}+\cdots+\alpha_1^{(M-1)})\ \cdot\ g\ \left(\underbrace{\begin{matrix} 1 & \\ 0 & \vec{\alpha}^{\,(1)}\cdots\vec{\alpha}^{\,(M-1)} \\ \vdots & \\ 0 & \end{matrix}}_{=:\ [1]}\right)\ ,$$

e.g., contains (in next to lowest order in λ) a term $\alpha_1^{(1)}\cdot\alpha_1^{(2)}\cdot[1]$ (of total degree $(M+1)$ in the $\alpha_s^{(\bar{r})}$), which cannot appear anywhere else (in the same order in λ), – in contradiction to the assumption that (57) should hold for $[\cdots]_\lambda$ (cp. (58)) replacing the curly bracket (4).

6. A Remark on Generalized Schild Actions

Consider

$$S \;:=\; -\int d\varphi^0\, d^M\varphi\, f(G)\;, \tag{62}$$

where $G := (-)^M \det\,(G_{\alpha\beta})$, $G_{\alpha\beta} := \frac{\partial x^\mu}{\partial\varphi^\alpha}\frac{\partial x^\nu}{\partial\varphi^\beta}\,\eta_{\mu\nu}$, $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,\cdots,-1)$, $\alpha,\beta = 0,\cdots,M$ and f some smooth monotonic function like G^γ ($\gamma = 1$ resp. $\frac{1}{2}$ corresponding to a generalized Schild-, resp. Nambu-Goto, action for M-branes). Apart from a few subtleties (like $\gamma = 1$ allowing for vanishing G, while $\gamma = \frac{1}{2}$ does not) actions with different f are equivalent, in the sense that the equations of motion,

$$\partial_\alpha\,\left(f'(G)G\,G^{\alpha\beta}\,\partial_\beta\,x^\mu\right) \;=\; 0 \qquad \mu \;=\; 0\cdots\cdot D-1 \tag{63}$$

are easily seen to imply

$$\partial_\alpha\,G \;=\; 0 \qquad \alpha \;=\; 0,\cdots,M \tag{64}$$

(just multiply (63) by $\partial_\epsilon x_\mu$ and sum) – unless $f(G) = \text{const.}\ \sqrt{G}$, in which case (62) is fully reparametrisation invariant and a parametrisation may be assumed in which $G =$ const. (such that (63) becomes proportional to $\partial_\alpha(G^{\alpha\beta}\,\partial_\beta x^\mu)$ also in this case). Due to

$$G \;=\; \sum_{\mu_1<\cdots<\mu_{M+1}} \{x^{\mu_1},\cdots,x^{\mu_{M+1}}\}\,\{x_{\mu_1},\cdots,x_{\mu_{M+1}}\} \tag{65}$$

(63) may be written as (cp. [14] for strings, and [15] for membranes, in the case of $\gamma = 1$ resp. $\frac{1}{2}$)

$$\left\{f'(G)\{x^{\mu_1},\cdots,x^{\mu_{M+1}}\},\; x_{\mu_2},\cdots,x_{\mu_{M+1}}\right\} \;=\; 0\;, \tag{66}$$

whose deformed analogue (note the similarity between $G =$ const. and condition (3.9) of [16])

$$\left[[x^{\mu_1},\cdots,x^{\mu_{M+1}}],\; x_{\mu_2},\cdots,x_{\mu_{M+1}}\right] \;=\; 0 \tag{67}$$

looks very suggestive when thinking about space-time quantization in M-brane theories.

7. Multidimensional Integrable Systems from M-algebras

Several ideas used in the context of integrable systems are based on bilinear operations. Our problems to extend results about low (especially 1+1) dimensional integrable field theories to higher dimensions may well rest on precisely this fact. Already some time ago, attempts were made to overcome this difficulty by generalizing Lax-pairs to -triples ([3], p. 72) and Hirota's bilinear equations for 'τ-functions' [17] to multilinear equations ([3], p. 107-111).

At that time, good examples were lacking, and – not being an exception to the rule that generalisations involving the number of dimensions (of one sort or an other) are usually hindered by implicitely low dimensional point(s) of view – the proposed generalisation of

Hirota-operators may have still been too naive; while hoping to come back to the question of multidimensional τ-functions in the near future, I would now like to give an example ($M > 3$ will then be obvious) for an equation of the form

$$\dot{\mathcal{L}} = \frac{1}{\rho}\{\mathcal{L},\ \mathcal{M}_1,\ \mathcal{M}_2\} \tag{68}$$

being equivalent to the equations of motion of a compact 3 dimensional manifold $\widehat{\sum} \subset \mathbb{R}^4$ (described by a time-dependent 4-vector $x^i(\varphi^1, \varphi^2, \varphi^3, t)$), moving in such a way that its normal velocity is always equal to the induced volume density $\sqrt{g}$ (on $\widehat{\sum}$) devided by a fixed non-dynamical density $\rho(\varphi)$ ('the' volume density of the parameter manifold):

$$\begin{aligned}
\dot{x}_1 &= \frac{1}{\rho}\{x_2, x_3, x_4\} \\
\dot{x}_2 &= -\frac{1}{\rho}\{x_3, x_4, x_1\} \\
\dot{x}_3 &= \frac{1}{\rho}\{x_4, x_1, x_2\} \\
\dot{x}_4 &= -\frac{1}{\rho}\{x_1, x_2, x_3\}\ .
\end{aligned} \tag{69}$$

With the curly bracket defined as before (cp. (4)), it will be an immediate consequence of (68) that

$$Q_n := \int_{\Sigma} d^3\varphi\ \rho(\varphi)\ \mathcal{L}^n \tag{70}$$

is time-independent (for any n).

In [2] evolution-equations of the form (69) (in any number of dimensions) were shown to correspond to the diffeomorphism invariant part of an integrable Hamiltonian field theory (as well as to a gradient flow); one way to solve such equations is to note ([18], [2]) that the time at which the hypersurface will pass a point $\vec{x}$ in space will simply be a harmonic function.

In any case, the (a) form of $(\mathcal{L}, \mathcal{M}_1, \mathcal{M}_2)$ that will yield the equivalence of (69) with (68) is:

$$\begin{aligned}
\mathcal{L} &= (x_1 + ix_2)\frac{1}{\lambda} + (x_3 + ix_4)\frac{1}{\mu} + \mu(x_3 - ix_4) - \lambda(x_1 - ix_2) \\
\mathcal{M}_1 &= \frac{\mu}{2}(x_3 - ix_4) - \frac{1}{2\mu}(x_3 + ix_4) \\
\mathcal{M}_2 &= \frac{\lambda}{2}(x_1 - ix_2) + \frac{1}{2\lambda}(x_1 + ix_2)
\end{aligned} \tag{71}$$

(involving two spectral parameters, λ and μ). Surely, this observation will have much more elegant formulations, and conclusions.

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References

[**1**] Y. Nambu; Phys. Rev. *D7* # 8 (1973) 2405.

[**2**] M. Bordemann, J. Hoppe; 'Diffeomorphism Invariant Integrable Field Theories and Hypersurface Motions in Riemannian Manifolds'; ETH-TH/95-31, FR-THEP-95-26.

[**3**] J. Hoppe; 'Lectures on Integrable Systems'; Springer-Verlag 1992.

[**4**] J. Hoppe; 'Quantum Theory of a Massless Relativistic Surface'; MIT Ph. D. thesis 1982 and Elem. Part. Res. J. (Kyoto) *80* (1989) 145.

[**5**] L. Takhtajan; Comm. Math. Phys. *160* (1994) 295.
R. Chatterjee; 'Dynamical Symmetries and Nambu Mechanics'; Stony Brook preprint 1995.
R. Chatterjee, L. Takhtajan; 'Aspects of Classical and Quantum Nambu Mechanics' (1995; to appear in Lett. Math. Phys.).

[**6**] V.T. Filippov; 'n-ary Lie algebras'; Sibirskii Math. J. *24* # 6 (1985) 126 (in russian).

[**7**] P. Lecomte, P. Michor, A. Vinogradov; 'n-ary Lie and Associative Algebras'; preprint 1994.

[**8**] H. Weyl; 'The Classical Groups'; 2$^{\text{nd}}$ edition, Princeton University Press.

[**9**] N. White; 'Theory of Matroids'; Cambridge University Press 1987.

[**10**] J.L. Loday; 'La renaissance des opérades'; in Séminaire Bourbaki, exposé 792, Novembre 1994.

[**11**] V. Ginzburg, M.M. Kapranov; Duke Math. J. *76* (1994) 203.

[**12**] Ph. Hanlon, M. Wachs; 'On Lie k-Algebras; preprint 1993.

[**13**] A.V. Gnedbaye; C.R. Acad. Sci. Paris, *t. 321*, Série I, p. 147, 1995.

[**14**] A. Schild; Phys. Rev. *D16* (1977) 1722.

[**15**] A. Sugamoto; Nucl. Phys. *B215* [FS7] (1983) 381.

[**16**] S. Doplicher, K. Fredenhagen, J. Roberts; Comm. Math. Phys. *172* (1995) 187.

[**17**] R. Hirota; 'Direct methods of finding solutions of nonlinear evolution equations', in Lect. Notes in Math. *515*, Springer-Verlag 1976.

[**18**] J. Hoppe; Phys. Lett. *B335* (1994) 41.

[**19**] P. Lecomte, C. Roger; 'Rigidité de l'algèbre de Lie des champs de vecteurs unimodulaires'; Université IGD Lyon 1, preprint (1995).